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Viscosity renormalization based on direct-interaction closure

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Approximations in statistical turbulence theory often rely on modelling the decay in time of velocity correlations with a simple exponential decay. The decay rate is viewed as a renormalized viscosity. The three simplest implementations of this approximation scheme were originally given independently by Kraichnan, Edwards and Leslie. Each of these investigators used a different formalism and each achieved different renormalization prescriptions. These three different results are reexamined here entirely in terms of direct-interaction theory. The difference in the prescriptions of Kraichnan and Leslie is shown to be the product of different definitions of renormalized viscosity. Edwards' prescription is shown to result from an inconsistent identification of the non-stationary energy-spectrum relaxation rate with the viscosity. An assessment of the validity of the Markovian closure approximation, and a prescription for non-stationary renormalized viscosity are provided.

1. Introduction

Statistical perturbation theories of turbulence have produced many different prescriptions for eddy viscosity, that is, renormalized molecular viscosity. Our object here is to give a precise self-consistent interpretation of the three prescriptions due originally to Edwards (1964), Kraichnan (1964) and Leslie (1973) respectively. These prescriptions derived for fully isotropic, homogeneous, incompressible flow take the form

$$\mu_k = \nu_k + \frac{2k^2}{d-1}\int \mathrm{d}^d p\, \mathrm{d}^d q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}) \frac{b_{k,p,q}\, U_q}{\epsilon\mu_k+\mu_p+\mu_q}, \tag{1.1}$$

where $\epsilon = \pm 1, 0$ differentiates between the three prescriptions. The vectors $\boldsymbol{k}, \boldsymbol{p}, \boldsymbol{q}$ are wavevectors arising from the spatial Fourier transform of the velocity field, which we take as

$$\boldsymbol{v}(\boldsymbol{x}, t) = \int \mathrm{d}^d k\, \mathrm{e}^{+\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}\, \boldsymbol{v}_{\boldsymbol{k}}(t).$$

d is the number of spatial dimensions, which we take to be two or three, although analytic continuation to arbitrary dimension $d \geqslant 2$ causes no difficulty (Fournier & Frisch 1978). U_k is the ensemble-average modal-energy spectrum, and the interaction coefficient $b_{k,p,q}$ is defined for dimension d by

$$b_{k,p,q} = \frac{1}{2k^2}\left(\frac{\sin\alpha}{k}\right)^2 [(p^2-q^2)(k^2-q^2)+(d-2)k^2p^2], \tag{1.2}$$

with α the angle between $\boldsymbol{p}$ and $\boldsymbol{q}$. ν_k is the linear viscosity, usually taken as the Laplacian form νk^2, but more general forms are admissible; μ_k is the renormalized viscosity, and $\mu_k - \nu_k$ is the eddy viscosity.

The $\epsilon = +1$ prescription was originally derived by Edwards (1964) by using a perturbation method in Fokker–Planck formalism. The $\epsilon = 0$ prescription was originally derived by Kraichnan (1964) using direct-interaction (DI) theory, and later by Herring (1965) using Fokker–Planck formalism. The $\epsilon = -1$ prescription was originally derived by Leslie (1973) using response-function Fokker–Planck formalism. For a review of these results see Leslie (1973) and Lee (1974).

There remains some confusion in the literature in understanding how such different prescriptions arise from derivations which all have similar physical assumptions. We hope to clear up this confusion by examining carefully the approximations necessary to obtain these prescriptions. We find it convenient to work entirely in DI theory (Kraichnan 1959), although a similar analysis can be given in Fokker–Planck perturbation theory. We feel that these investigations are easier to describe in DI theory, and, as Leslie (1973) has shown, DI theory is more general than the Fokker–Planck perturbation methods and consequently can be used to generate more precise albeit more complicated descriptions of turbulence. Also, we aim at directing some attention at the much neglected $\epsilon = -1$ prescription of Leslie (1973).

There are two essential approximations necessary to obtain renormalization approximations of the form (1.1). The first invokes a fluctuation–dissipation relation, which provides a connection between two-time velocity correlations and the response of the fluid to infinitesimal external perturbations. The second assumes that the response function is dominated by simple exponential decay in time. Carefully examining these approximations leads to the conclusion that the $\epsilon = +1$ prescription is inconsistent. We find that the right-hand side of (1.1) with $\epsilon = +1$ defines the rate γ_k at which the energy spectrum relaxes when perturbed from a stationary state, and we show that this is valid only when $\mu_k \gg \gamma_k$, so that it is inappropriate to identify μ_k with γ_k. The $\epsilon = 0$ prescription is shown to give a consistent approximation for the integral over all time of the response function as $1/\mu_k$, while the $\epsilon = -1$ prescription is shown to be a consistent approximation to the long-time response of the system.

In §2, we introduce the DI equations. Section 3 discusses the fluctuation–dissipation relation and its extension to non-stationary statistics. Section 4 examines the three prescriptions in stationary state. Section 5 deals with the non-stationary state and shows the inconsistency in the $\epsilon = +1$ prescription and the extension of the prescriptions to non-stationary flow. An interesting byproduct of this analysis is a better understanding of the regime of validity for Markovian closure theories of the eddy-damped quasinormal type.

DI theory is known to give inaccurate results for strong turbulence. Alternative theories which address the problem of high-Reynolds-number turbulence have been developed. These lead to renormalization prescriptons which do not simply derive from DI theory (cf. Kraichnan 1964, 1971, 1976; Leslie 1973; André 1974; Legras 1980). We attempt no analysis of those results here. For problems of weak turbulence, in particular in systems with a strong component of wave propagation, DI theory should be perfectly adequate. Thus it is important to understand clearly approximations to the DI equations, a point emphasized by some recent work in plasma and geophysical problems (cf. Holloway & Hendershott 1977; DuBois & Rose 1981; DeWitt & Wright 1982; Carnevale & Martin 1982; Carnevale & Frederiksen 1983).

2. Direct-interaction equations

We take as our starting point the DI equations. DI theory is a fairly general second-order renormalized perturbation theory, which can be applied to a wide class of field equations including the Navier–Stokes equation. The derivation and validity of the DI equations has been discussed at length in the literature. We refer the reader to Leslie (1973) and Martin, Siggia & Rose (1974) for the derivation and further references. Specializing to homogeneous isotropic incompressible turbulence, DI gives a coupled pair of integro-differential equations for the two-time ensemble-average correlation function of velocity amplitudes

$$\langle v_i(\mathbf{k},t)\, v_j(-\mathbf{k},t')\rangle = U_{ij}(\mathbf{k},t,t') = P_{ij}(\mathbf{k})\, U_k(t,t') \tag{2.1}$$

and the ensemble-average retarded response function

$$G_{ij}(\mathbf{k},t,t') = P_{ij}(\mathbf{k})\, G_k(t,t'), \tag{2.2}$$

where $v_i(\mathbf{k},t)$ is the ith Cartesian component of velocity amplitude and

$$P_{ij}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{k^2}.$$

$G_{ij}(\mathbf{k},t,t')$ is the spatial Fourier transform of the ensemble average of the retarded infinitesimal-impulse response function which gives the response of the field $v_i(\mathbf{x})$ at the time t due to an infinitesimal perturbation in the field $v_j(\mathbf{x}')$ at time t'. The term retarded indicates that this response function vanishes for $t < t'$ following the dictates of causality. Furthermore, by definition we must have $G_k(t,t) = 1$.

The ensemble for the average can be considered the ensemble of initial conditions. In DI theory, it is easy to include the effects of a random, Gaussian-distributed external forcing $f_i(\mathbf{k},t)$, which we add here for completeness. Hence the ensemble average also implies an average over the forcing distribution. We write the variance of the forcing as

$$\langle f_i(\mathbf{k},t) f_j(-\mathbf{k},t)\rangle = P_{ij}(\mathbf{k})\, F_k(t,t'). \tag{2.3}$$

The DI equations are

$$\left(\frac{\partial}{\partial t} + \nu_k\right) U_k(t,t') = \int_{t_0}^{\infty} \mathrm{d}t'' \,[\Sigma_k^{\mathrm{r}}(t,t'')\, U_k(t',t'') + \Sigma_k^{<}(t,t'')\, G_k(t',t'') + F_k(t,t'')\, G_k(t',t'')], \tag{2.4}$$

$$\left(\frac{\partial}{\partial t} + \nu_k\right) G_k(t,t') - \int_{t_0}^{\infty} \mathrm{d}t''\, \Sigma_k^{\mathrm{r}}(t,t'')\, G_k(t'',t') = \delta(t-t'). \tag{2.5}$$

where t_0 denotes the initial time ($t > t_0$ and $t' > t_0$). The Σ-functions are given by

$$\Sigma_k^{<}(t,t') = \frac{2k^2}{d-1} \int \mathrm{d}^d p\, \mathrm{d}^d q\, \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\, a_{kpq}\, U_p(t,t')\, U_q(t,t'), \tag{2.6}$$

$$\Sigma_k^{\mathrm{r}}(t,t') = -\frac{2k^2}{d-1} \int \mathrm{d}^d p\, \mathrm{d}^d q\, \delta(\mathbf{k}-\mathbf{p}-\mathbf{q})\, b_{kpq}\, G_p(t,t')\, U_q(t,t'), \tag{2.7}$$

where the coefficient b_{kpq} is given by (1.2) and the coefficient a_{kpq} is given by

$$a_{kpq} = \tfrac{1}{2}(b_{kpq} + b_{kqp}).$$

The notation $\Sigma^{<}$ and Σ^{r} is a holdover from quantum statistical theory (e.g. Langreth 1975). We note that the symmetry $U_k(t,t') = U_k(t',t)$ implies the same symmetry for $\Sigma_k^{<}(t,t')$, and the vanishing of $G_k(t,t')$ for $t < t'$ implies that $\Sigma_k^{\mathrm{r}}(t,t')$

vanishes for $t < t'$. Equations (2.4) and (2.5) are in the form of the Martin, Siggia & Rose (1974) equations, which represent a formally exact set of equations for G and U. In the Martin, Siggia & Rose (1974) equations the Σ are infinite series involving terms of higher and higher degree in the G and U, and these equations are the analogues of Dyson's equations for classical statistical field theory (cf. Roman 1969). DI theory truncates these series at the first terms, which are given by (2.6) and (2.7). Kraichnan (1961) has shown the DI equations to be exact for certain model problems. Again we refer the reader to the literature for a full critique of DI theory.

An equation for the evolution of the energy spectrum $U_k(t,t)$ can be obtained quite simply from (2.4). To do this we first note that

$$\frac{\partial U_k(t,t)}{\partial t} = \lim_{t'\to t} \frac{\partial U_k(t,t')}{\partial t} + \frac{\partial U_k(t,t')}{\partial t'} = \lim_{t'\to t} \frac{\partial U_k(t,t')}{\partial t} + \frac{\partial U_k(t',t)}{\partial t'},$$

where the limit is taken with $t' < t$ or $t' > t$. We also specialize to the case of white-noise forcing, that is,

$$F_k(t,t') = \delta(t-t')\,F_k, \tag{2.8}$$

because this is all we shall need here. Extensions to non-white forcing are more complicated, but can be handled in a straightforward way. The result for white forcing is

$$\left(\frac{\partial}{\partial t} + 2\nu_k\right) U_k(t,t) = F_k + 2\int_{t_0}^{\infty} dt''\,[\Sigma_k^{\text{T}}(t,t'')\,U_k(t,t'') + \Sigma_k^{<}(t,t'')\,G_k(t,t'')]. \tag{2.9}$$

3. Non-stationary fluctuation–dissipation relation

An essential step in obtaining the prescriptions considered here is the simplification of the DI equations through the introduction of a fluctuation–dissipation relation. This relation permits the elimination of $U_k(t,t')$ in terms of $G_k(t,t')$. It has been proved (Kraichnan 1959*b*; Leith 1975) that for unforced inviscid spectrally truncated flow in stationary state, the fluctuation–dissipation relation

$$U_k(t,t')\,\theta(t-t') = G_k(t,t')\,U_k^{\text{st}}, \tag{3.1}$$

holds exactly. Here $\theta(t-t')$ is the Heaviside step function and U_k^{st} is the stationary energy spectrum. For unforced inviscid spectrally truncated flow U_k^{st} has the canonical equilibrium form (Kraichnan 1975)

$$U_k^{\text{st}} = (\alpha + \beta k^2)^{-1}, \tag{3.2}$$

where α and β are constants with $\beta = 0$ in more than two dimensions ($d > 2$). Also for forced viscid dynamics where the stationary-state probability distribution is Gaussian, the fluctuation–dissipation relation (3.1) can be shown to be exact (Deker & Haake 1975). For example, this would be the case for Thompson's (1972) solution for spectrally truncated flow with white-noise forcing with spectrum

$$F_k = 2\nu_k(\alpha + \beta k^2)^{-1}. \tag{3.3}$$

Furthermore, there is some evidence to suggest that (3.1) is a reasonable approximation to realistic turbulent flows except in the dissipation range (Leith 1975; Herring & Kraichnan 1972; Bell 1980).

In order to obtain renormalization prescriptions that are valid in nonstationary states and also to understand the Edwards prescription ($\epsilon = +1$) fully, it is necessary to generalize the fluctuation–dissipation relation to non-stationary states.

It follows directly from the DI equations that for white-noise forcing

$$U_k(t,t')\,\theta(t-t') = G_k(t,t')\,U_k(t',t') + \int_{t'}^{t} \mathrm{d}t'' \int_{t_0}^{t'} \mathrm{d}t'''\,G_k(t,t'')\,[\Sigma_k^{>}(t'',t''')\,U_k(t''',t') + \Sigma_k^{<}(t'',t''')\,G_k(t',t''')]. \tag{3.4}$$

The details of the derivation of this relation are given in the appendix.

There are certain cases for which the integral on the right-hand side of (3.4) vanishes exactly. For example, this is true for the stationary states mentioned above (Thompson 1972; Kraichnan 1975). This is also true for certain simple non-stationary states (Thompson 1982; Carnevale 1982). Near such states, the neglect of this integral should be a valid approximation.

In general, it is difficult to assess *a priori* how large an error is made by neglecting this integral. We can again cite the work of Leith (1975), Herring & Kraichnan (1972) and Bell (1980) as partial justification for such an approximation in realistic turbulence applications. As we shall see, this approximation is necessary to obtain simple viscosity renormalization. By neglecting the integral term we obtain the non-stationary fluctuation–dissipation relation in the form

$$U_k(t,t')\,\theta(t-t') = G_k(t,t')\,U_k(t',t'). \tag{3.5}$$

This form is also given by Orszag (1970), Lee (1974) and others. The significant thing to note is that the energy spectrum on the right-hand side of (3.5) is evaluated at t' and not t. This will be crucial in our analysis of Edwards' prescription ($\epsilon = +1$). Note that (3.5) is the appropriate generalization of the fluctuation–dissipation relation independent of whether or not the system is nonlinear.

This point is sufficiently important that we take space here to emphasize it with a simple example. Consider the following simple Langevin equation (cf. Chandrasekhar 1943; Leith 1971):

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} + \lambda\right) a(t) = f(t). \tag{3.6}$$

We take all quantities as real, although the extension to the complex domain is also easy. We assume the initial state ensemble is such that

$$\langle a(t_0)\rangle = 0, \quad \langle a^2(t_0)\rangle = a_0^2,$$

and $f(t)$ is taken as randomly distributed and white noise in time with variance

$$\langle f(t)f(t')\rangle = F\delta(t-t'), \quad \langle a(t_0)f(t)\rangle = 0.$$

The statistics are easily computed exactly for all time with the following relevant results:

$$\langle a(t)\,a(t')\rangle\,\theta(t-t') = \frac{F}{2\lambda}\exp\left[-\lambda(t-t')\right]\theta(t-t') + \left(a_0^2 - \frac{F}{2\lambda}\right)\exp\left[-\lambda(t+t'-2t_0)\right]\theta(t-t'), \tag{3.7}$$

$$\langle a^2(t')\rangle = \frac{F}{2\lambda} + \left(a_0^2 - \frac{F}{2\lambda}\right)\exp\left[-2\lambda(t'-t_0)\right]. \tag{3.8}$$

The response function for this problem is trivially given by the linear retarded Green function

$$G(t,t') = \exp\left[-\lambda(t-t')\right]\theta(t-t'). \tag{3.9}$$

By combining these results we see that the non-stationary fluctuation–dissipation relation

$$\langle a(t)\, a(t')\rangle\, \theta(t-t') = G(t,t')\, \langle a^2(t')\rangle \tag{3.10}$$

is exact at all times, no matter how far from the stationary state.

For this simple linear system, the decorrelation rate λ of the two-time correlation and response functions and the relaxation to equilibrium rate 2λ of the single-time correlation function are simply related. However, for nonlinear systems this simple relationship is not in general valid (Leith 1971, appendix §*f*).

4. Stationary state

We begin our analysis of viscosity renormalization with a study of the stationary-state DI equations. In stationary state the correlation and response functions can be written in terms of a single temporal variable $t-t'$, that is, $U_k(t-t')$ and $G_k(t-t')$. The initial time t_0 can be taken as $-\infty$, and we can define frequency transforms according to

$$U_k(\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\omega\tau}\, U_k(\tau)\,\mathrm{d}\tau. \tag{4.1}$$

For simplicity we use the same symbol for a function and its Fourier transform since the nature of the argument adequately differentiates between the two. The response-function equation (2.5) then can be written as

$$G_k(\omega) = \frac{\mathrm{i}}{\omega+\mathrm{i}\nu_k-\mathrm{i}\Sigma_k^{\mathrm{T}}(\omega)}, \tag{4.2}$$

and, with the stationary fluctuation–disspation relation (3.1), we have

$$\Sigma_k^{\mathrm{T}}(\omega) = \frac{-k^2}{\pi(d-1)}\int \mathrm{d}^d p\, \mathrm{d}^d q\, \mathrm{d}\omega_1\, \mathrm{d}\omega_2\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\, \delta(\omega-\omega_1-\omega_2)\, b_{kpq}\, U_q^{\mathrm{st}} G_p(\omega_1)\, G_q(\omega_2). \tag{4.3}$$

The linear response function is simply an exponential in time:

$$G_k^0(t-t') = \mathrm{e}^{-\nu_k(t-t')}\,\theta(t-t'), \tag{4.4}$$

or equivalently a simple pole in the complex ω-plane:

$$G_k^0(\omega) = \frac{\mathrm{i}}{\omega+\mathrm{i}\nu_k}. \tag{4.5}$$

For the nonlinear system we suppose that $G_k(\omega)$ in general has poles and branch cuts in the complex ω-plane (Roman 1969; DuBois & Rose 1981); moreover, one of the poles at say $\omega = z_k = -\mathrm{i}\mu_k$ is the nonlinear modification of the pole $\omega = -\mathrm{i}\nu_k$ of the linear response function. From (4.2) it then follows that the *exact* equation for the singularity z_k is

$$z_k = -\mathrm{i}\nu_k + \mathrm{i}\Sigma_k^{\mathrm{T}}(z_k); \tag{4.6}$$

this corresponds to mass renormalization of quantum field theory (Roman 1969, equations (5–97)). Thus the renormalized viscosity must satisfy the *exact* equation

$$\mu_k = \nu_k - \Sigma_k^{\mathrm{T}}(-\mathrm{i}\mu_k). \tag{4.7}$$

It must be emphasized that this equation does not depend for its validity on any simple pole approximation to the response function. For actually evaluating $\Sigma_k^{\mathrm{T}}(-\mathrm{i}\mu_k)$ we shall, however, use a single-parameter representation of $G_k(\omega)$ in which ν_k in (4.4) and (4.5) is replaced by the renormalized viscosity μ_k. The self-energy $\Sigma_k^{\mathrm{T}}(\omega)$

is a function analytic in the upper half-plane; thus its evaluation at $\omega = -\mathrm{i}\mu_k$ needs to be accomplished through analytic continuation into the lower half-plane. This continuation is most readily carried out for the corresponding discrete problem where the continuous integrals over $\boldsymbol{p}$ and $\boldsymbol{q}$ are replaced by discrete sums. This would certainly be an appropriate description of computer simulations, and is in accord with the notion of approximating real flows by taking smaller and smaller mesh sizes in a discrete representation of physical space. With discrete sums we see that if we substitute in a simple-pole form for $G(\omega)$ in $\Sigma(\omega)$, then the analytic continuation of $\Sigma(\omega)$ off the real axis causes no difficulty, and finding the singularities of (4.2) reduces to finding the roots of a polynomial. Thus we are led to a continued-fraction or Padé approximation for $G(\omega)$. To make this scheme explicit we must first examine the behaviour of $G(\omega)$ near a simple pole a little more carefully. Let $z_k^{(j)}$ be a root of (4.6). Then we have by Taylor expansion of $\Sigma_k^{\mathrm{T}}(\omega)$

$$G_k(\omega \approx z_k^{(j)}) = \frac{\mathrm{i}}{\omega + \mathrm{i}\nu_k - \mathrm{i}\left(\Sigma_k^{\mathrm{T}}(z_k^{(j)}) + (\omega - z_k^{(j)})\left.\dfrac{\partial \Sigma_k^{\mathrm{T}}}{\partial \omega}\right|_{z_k^{(j)}} + O(\omega - z_k^{(j)})^2\right)}$$

$$= \frac{\mathrm{i}}{(\omega - z_k^{(j)})\left(1 - \mathrm{i}\left.\dfrac{\partial \Sigma_k^{\mathrm{T}}}{\partial \omega}\right|_{z_k^{(j)}}\right) + O(\omega - z_k^{(j)})^2}.$$

Thus we see that, provided that

$$1 - \mathrm{i}\left.\frac{\partial \Sigma_k^{\mathrm{T}}}{\partial \omega}\right|_{z_k^{(j)}} \neq 0,$$

we have a simple pole at $z_k^{(j)}$ with residue

$$\mathrm{i}Z_k^{(j)} = \mathrm{i}\left(1 - \mathrm{i}\left.\frac{\partial \Sigma_k^{\mathrm{T}}}{\partial \omega}\right|_{z_k^{(j)}}\right)^{-1}. \tag{4.8}$$

This leads us to an approximate representation of $G_k(\omega)$ as a sum of simple poles with positions given by $z_k^{(j)}$ and residues $\mathrm{i}Z_k^{(j)}$ or equivalently $G_k(t-t')$ a sum of exponentials with weights $Z_k^{(j)}$. The implementation of such an approximation scheme would be iterative and would clearly be applicable to systems with linear wave modes as well as viscosity. This suggestion for fluids has been given by Carnevale & Martin (1982) for β-plane turbulence, Carnevale & Frederiksen (1983) for internal waves, and implemented by DuBois & Rose (1981) for Langmuir turbulence and by DeWitt & Wright (1982) for internal waves. There is evidence to suggest that branch cuts in $G(\omega)$ can also be studied by this method, by iterating to high numbers of poles (DuBois & Rose 1981; Kraichnan 1970; Common 1970).

To make a connection with single-parameter representations of $G_k(\omega)$ we focus on that root $z_k = -\mathrm{i}\mu_k$ which is the continuation of $-\mathrm{i}\nu_k$ in the linear case, that is, the root that reduces to $-\mathrm{i}\nu_k$ in the limit $U_k^{\mathrm{st}} \to 0$. Then μ_k satisfies (4.7) and

$$G_k(\omega \approx -\mathrm{i}\mu_k) \approx \frac{\mathrm{i}Z_k}{\omega + \mathrm{i}\mu_k}, \tag{4.9}$$

with

$$Z_k = \left(1 - \mathrm{i}\left.\frac{\partial \Sigma_k^{\mathrm{T}}}{\partial \omega}\right|_{\omega = -\mathrm{i}\mu_k}\right)^{-1}. \tag{4.10}$$

If we wish to neglect all contributions to $G_k(\omega)$ except this effect of nonlinear correction to viscosity, we cannot simply replace $G_k(\omega)$ by (4.9) because this does not satisfy the condition $G_k(t, t') = 1$, or equivalently the sum rule

$$\int G_k(\omega)\,\mathrm{d}\omega/2\pi = 1.$$

Thus to replace $G_k(\omega)$ by simply the part which represents an effective viscosity we define the renormalized propagator

$$G_k(\omega) = \frac{\mathrm{i}}{\omega + \mathrm{i}\mu_k}, \tag{4.11}$$

or equivalently

$$G_k(t-t') = \exp[-\mu_k(t-t')]\,\theta(t-t'). \tag{4.12}$$

This is the analogue of propagator renormalization in quantum field theory (cf. Roman 1969). Calculating $\Sigma^{\mathrm{T}}_k(\omega)$ with (4.11) yields

$$\Sigma^{\mathrm{T}}_k(\omega) = -\frac{2k^2}{d-1}\int \mathrm{d}^d p\,\mathrm{d}^d q\,\delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\frac{b_{kpq}U^{\mathrm{st}}_q}{-\mathrm{i}\omega+\mu_p+\mu_q}. \tag{4.13}$$

Again we note that, for a discrete representation of the integrals in (4.13), the analytic continuation of ω into the lower half-plane causes no difficulty. So, if we evaluate $\Sigma(\omega)$ at $\omega = -\mathrm{i}\mu_k$, equation (4.7) for μ_k becomes

$$\mu_k = \nu_k + \frac{2k^2}{d-1}\int \mathrm{d}^d p\,\mathrm{d}^d q\,\delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\frac{b_{kpq}U^{\mathrm{st}}_q}{-\mu_k+\mu_p+\mu_q}. \tag{4.14}$$

This is the same result that Leslie (1973) obtained from his Fokker–Planck response-function formulation, that is, the $\epsilon = -1$ prescription. Thus we see that the $\epsilon = -1$ prescription gives an approximation to the shift in the linear pole at $\omega = -\mathrm{i}\nu_k$ due to nonlinear effects.

If we consider the representation with continuous integrals over wavevectors, then the analytic continuation of $\Sigma^{\mathrm{T}}_k(\omega)$ into the lower half-plane is more difficult. $\Sigma^{\mathrm{T}}_k(\omega)$ as written in (4.13) is analytic in the upper half-plane but may have a complicated branch structure in the lower half-plane. Thus the evaluation of $\Sigma^{\mathrm{T}}_k(-\mathrm{i}\mu_k)$ is no longer straightforward. A complete treatment would require additional information about this branch structure or direct numerical calculation. A simpler approach would be to evaluate $\Sigma^{\mathrm{T}}_k(\omega)$ where it is analytic as close to the singularity $\omega = -\mathrm{i}\mu_k$ as possible, that is, at $\omega = 0$. This leads to the approximation

$$\mu_k = \nu_k - \Sigma^{\mathrm{T}}_k(0) \tag{4.15}$$

to the exact expression (4.7). Equation (4.15) with the simple-pole representation of $G_k(\omega)$ gives

$$\mu_k = \nu_k + \frac{2k^2}{d-1}\int \mathrm{d}^d p\,\mathrm{d}^d q\,\delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\frac{b_{kpq}U^{\mathrm{st}}_q}{\mu_p+\mu_q}. \tag{4.16}$$

Equation (4.15) is the analogue of intermediate renormalization in quantum physics (Bjorken & Drell 1965, §19.9). The result (4.16) is Kraichnan's (1964) prescription, that is, the $\epsilon = 0$ prescription. Note that

$$G_k(\omega = 0) = \int_0^\infty G_k(t-t')\,\mathrm{d}(t-t'),$$

$$G_k(\omega = 0) = \frac{1}{\nu_k - \Sigma_k(0)}.$$

We see that intermediate renormalization is equivalent to defining

$$\mu_k \equiv (G_k(\omega = 0))^{-1}. \tag{4.17}$$

Thus μ_k given by the $\epsilon = 0$ prescription actually gives an approximation to the full integral in time of the response function, and only indirectly an approximation of the nonlinear correction to ν_k.

Finally we turn to Edwards ($\epsilon = +1$) prescription. From (1.1) and (4.13) we immediately see that this prescription can be written as

$$\mu_k = \nu_k - \Sigma_k^{\mathrm{r}}(+\mathrm{i}\mu_k),$$

when the renormalized pole approximation for $G_k(\omega)$ is used. This would not seem to have any particular significance with respect to the response function equation. To see the significance of $\Sigma_k^{\mathrm{r}}(+\mathrm{i}\mu_k)$ we must go to the energy equation (2.9). For the stationary state with stationary fluctuation–dissipation relation, (2.9) reduces to

$$2\nu_k U_k^{\mathrm{st}} = F_k + 2\int_{-\infty}^{\infty} \mathrm{d}t'' \,[\Sigma_k^{\mathrm{r}}(t-t'')\, U_k^{\mathrm{st}} + \Sigma_k^{<}(t-t'')]\, G_k(t-t''), \tag{4.18}$$

or equivalently in terms of frequency transforms

$$2\nu_k U_k^{\mathrm{st}} = F_k + 2\int_{-\infty}^{\infty} \frac{\mathrm{d}\omega}{2\pi} [\Sigma_k^{\mathrm{r}}(-\omega)\, U_k^{\mathrm{st}} + \Sigma_k^{<}(-\omega)]\, G_k(\omega). \tag{4.19}$$

We now use the simple exponential form for $G_k(t-t')$ or equivalently the simple-pole form for $G_k(\omega)$ to obtain

$$2\nu_k U_k^{\mathrm{st}} = F_k + \frac{4k^2}{d-1}\int \mathrm{d}^d p\, \mathrm{d}^d q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}) \frac{a_{kpq} U_p^{\mathrm{st}} U_q^{\mathrm{st}} - b_{kpq} U_q^{\mathrm{st}} U_k^{\mathrm{st}}}{\mu_k + \mu_p + \mu_q}. \tag{4.20}$$

Equation (4.20) is the same result as obtained by Edwards (1964). We can rewrite this expression as

$$2(\nu_k - \Sigma_k^{\mathrm{r}}(\mathrm{i}\mu_k))\, U_k^{\mathrm{st}} = F_k + \Delta F_k. \tag{4.21}$$

This suggested to Edwards that one could identify $\nu_k - \Sigma_k^{\mathrm{r}}(\mathrm{i}\mu_k)$ as the renormalized viscosity μ_k. We believe this identification is erroneous and feel that the identification of this quantity as a relaxation rate

$$\gamma_k = \nu_k - \Sigma_k^{\mathrm{r}}(\mathrm{i}\mu_k) \tag{4.22}$$

is correct, as we shall demonstrate in §5. The relaxation rate γ_k is the rate at which a perturbation of the energy spectrum U_k^{st} relaxes, and should not be confused with renormalized viscosity, which parametrizes the response function. An interesting example which illustrates how misleading the stationary energy-balance equation may be is Thompson's (1972) exact two-dimensional stationary solution, with forcing defined by

$$2\nu_k U_k^{\mathrm{st}} = F_k, \tag{4.23}$$

with $U_k^{\mathrm{st}} = (\alpha + \beta k^2)^{-1}$. In that case, we have $\Sigma_k^{\mathrm{r}}(\mathrm{i}\mu_k)\, U_k^{\mathrm{st}} = -\frac{1}{2}\Delta F_k$. Even though the energy-balance equation assumes the simple linear form (4.23), we must still expect that the response function reflects the underlying nonlinear dynamics and cannot be represented by the linear response function.

5. Approach to stationary state

In order to give a physical interpretation to the relaxation rate defined by

$$\begin{aligned}\gamma_k &= \nu_k - \Sigma_k^{\mathrm{r}}(\mathrm{i}\mu_k) \\ &= \nu_k + \frac{2k^2}{d-1}\int \mathrm{d}^d p\, \mathrm{d}^d q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}) \frac{b_{kpq} U_q^{\mathrm{st}}}{\mu_k + \mu_p + \mu_q},\end{aligned} \tag{5.1}$$

we must examine the manner in which the single-time correlation function or energy spectrum approaches stationary state. The non-stationary energy equation for white-noise forcing is again

$$\left(\frac{\partial}{\partial t}+2\nu_k\right) U_k(t,t) = F_k+2\int_0^\infty \mathrm{d}t'' \,[\Sigma_k^{\eta}(t,t'')\, U_k(t,t'')+\Sigma_k^{\lessgtr}(t,t'')\, G_k(t,t'')], \tag{5.2}$$

where we have set $t_0 = 0$ for convenience. We now imagine that at $t = 0$ the energy spectrum is slightly perturbed from stationary state. We introduce the non-stationary fluctuation–dissipation relation (3.5) to obtain

$$\left(\frac{\partial}{\partial t}+2\nu_k\right) U_k(t,t)-F_k = 2\int_0^\infty \mathrm{d}t'' \,[\Sigma_k^{\eta}(t,t'')\, G_k(t,t'')\, U_k(t'',t'')+\Sigma_k^{\lessgtr}(t,t'')\, G_k(t,t'')]. \tag{5.3}$$

If we then assume that the response rate μ_k is much larger than the rate of change of the energy spectrum, we can simplify (5.3) even further. According to this slow-decay assumption, it becomes reasonable to consider an approximation to the following Taylor series:

$$G_k(t,t'')\, U_k(t'',t'') = G_k(t,t'')\left[U_k(t,t)+(t-t'')\frac{\partial U_k(t,t)}{\partial t}+\tfrac{1}{2}(t-t'')^2\frac{\partial^2 U(t,t)}{\partial t^2}+\dots\right]. \tag{5.4}$$

Since $G_k(t,t'')$ is significant only when $t-t''$ is small, and since we are assuming that $U_k(t,t)$ changes only very slowly, it may be appropriate to approximate (5.4) by

$$G_k(t,t'')\, U_k(t'',t'') \approx G_k(t,t'')\, U_k(t,t). \tag{5.5}$$

We can call (5.5) the Markovian approximation, for as we shall see it leads to the Markovian closure theory of turbulence. Using (5.5) in (5.3) yields

$$\left(\frac{\partial}{\partial t}+2\nu_k\right) U_k(t,t)-F_k = 2\int_0^\infty \mathrm{d}t'' \,[\Sigma_k^{\eta}(t,t'')\, U_k(t,t)+\Sigma_k^{\lessgtr}(t,t'')]\, G_k(t,t''). \tag{5.6}$$

By using the Markovian approximation (5.5) in writing out the Σ, using the renormalized propagator $G_k(t-t') = \theta(t-t')\exp[-\mu_k(t-t')]$, and performing the t'' integral, we obtain

$$\left(\frac{\partial}{\partial t}+2\nu_k\right) U_k(t,t)-F_k = \frac{4k^2}{d-1}\int \mathrm{d}^d p\, \mathrm{d}^d q\, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\frac{1-\exp[-(\mu_k+\mu_p+\mu_q)\,t]}{\mu_k+\mu_p+\mu_q}$$
$$\times [a_{kpq}\, U_p(t,t)\, U_q(t,t)-b_{kpq}\, U_q(t,t)\, U_k(t,t)]. \tag{5.7}$$

This is the eddy-damped quasinormal Markovian (EDQNM) closure (cf. Rose & Sulem 1978; Fournier & Frisch 1978). We have seen that it follows from the DI equations and the fluctuation–dissipation relation when $U_k(t,t)$ evolves sufficiently slowly. How slow is slow enough will become apparent in what follows.

We note that if we take $t \gg \mu^{-1}$, by which we mean that t is sufficiently large so that terms like $\exp[-(\mu_k+\mu_p+\mu_q)]\,t$ can be neglected relative to unity, then (5.7) can be written simply as

$$\left(\frac{\partial}{\partial t}+2(\nu_k-\Sigma_k^{\eta}(\mathrm{i}\mu_k))\right) U_k(t,t) = F_k+\Delta F_k. \tag{5.8}$$

This suggests that $2\gamma_k = 2(\nu_k-\Sigma_k^{\eta}(\mathrm{i}\mu_k))$ is the rate at which $U_k(t,t)$ approaches the stationary spectrum.

Let us now take this suggestion of the Markovian result seriously and determine under what circumstances it is consistent with the equations before the Markovian approximation is applied (i.e. (5.3)). Imagine a small perturbation $U_k^0 - U_k^{st}$ at a single wavenumber k to the energy spectrum at time $t = 0$. Then the simple-decay result suggests

$$U_k(t,t) = U_k^{st} + (U_k^0 - U_k^{st}) \exp(-2\gamma_k t) = U_k^{st} + \delta U_k(t). \tag{5.9}$$

Substituting (5.9) into (5.3) yields

$$\left(\frac{\partial}{\partial t} + 2\nu_k\right) \delta U_k(t) = -\frac{4k^2}{d-1} \int \mathrm{d}^d p \, \mathrm{d}^d q \, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}) \frac{b_{kpq} U_q^{st}}{\mu_k + \mu_p + \mu_q - 2\gamma_k} \times (1 - \exp[-(\mu_k + \mu_p + \mu_q - 2\gamma_k)t]) \, \delta U_k(t), \tag{5.10}$$

where we have used the definition of U_k^{st} as given by the stationary energy equation (4.21). If we assume that $2\gamma_k \leqslant \mu_k + \mu_p + \mu_q$ for all k, p and q, then (5.10) for large t (i.e. $t \gg \mu^{-1}$) gives the spectral relaxation rate as

$$\gamma_k = \nu_k + \frac{2k^2}{d-1} \int \mathrm{d}^d p \, \mathrm{d}^d q \, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}) \frac{b_{kpq} U_q^{st}}{\mu_k + \mu_p + \mu_q - 2\gamma_k}. \tag{5.11}$$

If we assume that $\gamma_k \ll \mu_k$, then (5.11) reduces to (5.1) as given by Lee (1974). On the other hand, if we try to impose $\gamma_k = \mu_k$, as Edwards' (1964) treatment suggests, then (5.11) gives

$$\mu_k = \nu_k + \frac{2k^2}{d-1} \int \mathrm{d}^d p \, \mathrm{d}^d q \, \delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}) \frac{b_{kpq} U_q^{st}}{-\mu_k + \mu_p + \mu_q}, \tag{5.12}$$

which is the Leslie (1973) ($\epsilon = -1$) result and *not* the Edwards (1964) ($\epsilon = +1$) result. Of course, there are sytems for which the relaxation rate of $U_k(t,t)$ is exactly twice the response rate. For example, this is the case for the linear example (3.6), and we note that Edwards' (1964) formalism is based on a direct analogy to the linear system. However, (5.10) indicates that we should not in general assume this to be the case for nonlinear dynamics (cf. Leith 1971).

We can now say something about the validity of the Markovian approximation. If we use the DI energy equation (2.9) with the non-stationary fluctuation–dissipation relation (3.5) and substitute in the renormalized propagator (4.12) and the exponentially decaying energy spectrum (5.9), we obtain the Markovian closure equations (5.7) up to terms of order γ/μ. By this we mean that if terms of the form $2\gamma_p/(\mu_k + \mu_p + \mu_q)$ are neglected relative to unity, then (5.7) results. This would indicate that if $U_k(t,t)$ approaches equilibrium exponentially with rate $2\gamma_k \ll \mu_k$, or slower than exponentially, then Markovian closure is a valid approximation to the DI energy equation.

Finally, we turn to the question of defining the response rate μ_k for the non-stationary case. This can be done in a systematic way when the Markovian approximation is valid (cf. Carnevale & Martin 1982). In that case, the response function $G_k(t,t')$ varies much more quickly in the difference variable $\tau = t - t'$ than in the sum variable $T = \frac{1}{2}(t+t')$. It is then reasonable to write the response function in terms of these slow and fast variables as

$$G_k(t,t') \equiv G_k(\tau; T),$$

and interpret it as the response of the system at time T in the evolution of the system.

We note that

$$\frac{\partial}{\partial t}=\frac{\partial}{\partial \tau}+\frac{1}{2}\frac{\partial}{\partial T},$$

and we can rewrite the response equation entirely in terms of τ and T. Then, if we Fourier-transform in the fast variable, we obtain

$$G_k(\omega;T)=\frac{\mathrm{i}}{\omega+\mathrm{i}\nu_k-\mathrm{i}\Sigma^{\mathrm{r}}_k(\omega;T)}+O\left(\frac{\partial U(T,T)}{\partial T}\right), \tag{5.13}$$

where now

$$\Sigma^{\mathrm{r}}_k(\omega;T)\equiv\frac{-2k^2}{d-1}\iint \mathrm{d}^d p\,\mathrm{d}^d q\,\delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\frac{b_{kpq}U_q(T,T)}{\mathrm{i}\omega+\mu_p+\mu_q}, \tag{5.14}$$

and $U_k(T,T)$ is the energy spectrum at time T. Thus neglecting terms $O(\partial U(T,T)/\partial T)$, or $O(\gamma/\mu)$ in our previous language, we can again define μ based on the ω-plane structure of $G_k(\omega;T)$; now, however, μ is taken to have a slow variation in time T. Thus in the Markovian approximation we have

$$\mu^0_k=\nu_k+\frac{2k^2}{d-1}\iint \mathrm{d}^d p\,\mathrm{d}^d q\,\delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\frac{b_{kpq}U_q(T,T)}{\mu^0_p+\mu^0_q}, \tag{5.15}$$

$$\mu_k=\nu_k+\frac{2k^2}{d-1}\iint \mathrm{d}^d p\,\mathrm{d}^d q\,\delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\frac{b_{kpq}U_q(T,T)}{-\mu_k+\mu_p+\mu_q}, \tag{5.16}$$

$$\gamma_k=\nu_k+\frac{2k^2}{d-1}\iint \mathrm{d}^d p\,\mathrm{d}^d q\,\delta(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\frac{b_{kpq}U_q(T,T)}{\mu_k+\mu_p+\mu_q}. \tag{5.17}$$

As we have discussed, μ^0_k is an approximation to the reciprocal of the integral over τ of $G_k(\tau;T)$ at time T, μ_k is an approximation to the shifted position of the linear pole at ν_k and should describe the large-τ behaviour of $G_k(\tau;T)$, and $2\gamma_k$ is the spectral relaxation rate in the approach to the stationary spectrum.

It is interesting to compare these results with the theory of quasiparticles in quantum statistical field theory. The quasiparticle is a collective oscillation with frequency or energy ω_k and lifetime μ_k. It is usually assumed that $\omega_k \gg \mu_k$, and the lifetime μ_k is given approximately by

$$\mu_k=-\operatorname{Re}\Sigma^{\mathrm{r}}_k(\omega_k). \tag{5.18}$$

The equation for the decay of the quasiparticle spectrum is written

$$\frac{\partial U_k(t,t)}{\partial t}=2U_k(t,t)\operatorname{Re}\Sigma^{\mathrm{r}}_k(\omega_k)+2\Sigma^{<}_k(\omega_k), \tag{5.19}$$

which is the analogue of our energy equation (5.8) (cf. Kadanoff & Baym 1964). So no discrepancy appears since the decay rate is the same as the relaxation rate given by (5.19) (i.e. what we have called γ_k). This is actually illusory because μ_k has been neglected relative to ω_k in these equations. An analysis of the quasiparticle picture in which μ_k is assumed finite and retained generalizes (5.18) to

$$\mu_k=-\operatorname{Re}\Sigma^{\mathrm{r}}_k(\omega_k-\mathrm{i}\mu_k), \tag{5.20}$$

and the relaxation rate in (5.19) becomes

$$\gamma_k=-\operatorname{Re}\Sigma^{\mathrm{r}}_k(\omega_k+\mathrm{i}\mu_k), \tag{5.21}$$

in direct analogy to our analysis (cf. Fetter & Walecka 1971). In fact, by a straightforward generalization of Leslie's prescription, Carnevale & Martin (1982) have shown that (5.20) and (5.21) hold for fluid problems in which the interaction between waves with frequency ω_k and turbulence is considered. That generalization also provides a prescription for frequency renormalization.

6. Discussion

We have presented an analysis of the three simplest viscosity-renormalization prescriptions which are based on a simple-pole approximation to the fluid response function in second-order closure. This analysis suggests that Edwards' (1964) identification of the energy-spectrum relaxation rate with eddy viscosity is inconsistent with DI theory. Kraichnan's (1964) prescription provides a self-consistent approximation to the time integral of the response function. Leslie's (1973) prescription gives an approximation to the position of the shifted viscous response pole in the complex-frequency plane.

We have emphasized the merits of Leslie's prescription. It is the analogue of quantum field theory renormalization and it is easily generalized to a multiple-pole representation of the response function. However, the possible vanishing of the denominator $-\mu_k+\mu_p+\mu_q$ may cause difficulty in the implementation of Leslie's prescription. Consider, for example, the case of periodic boundary conditions on a finite box, and a linear viscosity given by $\nu_k = \nu k^2$. The components of the wavevectors are integers, so solution by iteration will diverge if the initial approximation is $\mu_k = \nu_k$. This difficulty can be avoided by adding a small constant term to the initial approximation.† If the wavevector sums are approximated as integrals, then we have found in some model studies that the singularity is integrable on first iteration. If, however, inertial-range forms for U_k and μ_k are assumed, then the integral diverges. It is believed that this divergence, which also occurs in the Kraichnan and Edwards prescriptions, cannot be avoided within the strict confines of the Eulerian DI theory (cf. Leslie 1973).

The analytic structure of the self-energy in a continuous-wavevector description is far more complicated than the meromorphic case treated here. In a future paper we will say more on the possibilities in that situation.

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Appendix

The detailed derivation of (3.4) contained here follows that given by Martin *et al.* (1973). First we define

$$U_k^+(t,t') \equiv U_k(t,t')\,\theta(t-t'). \tag{A 1}$$

† Often in two-dimensional applications it is appropriate to include a Rayleigh friction in the linear viscosity (i.e. $\nu_k = \nu k^2 + R$, where R is independent of k). As long as R/ν is not an integer, the first iteration is well defined.

From (2.4) we obtain

$$\begin{aligned}\frac{\partial}{\partial t} U_k^{\mathrm{r}}(t,t') &= U_k(t,t')\,\delta(t-t') + \theta(t-t')\frac{\partial}{\partial t} U_k(t,t') \\ &= U_k(t,t')\,\delta(t-t') - \theta(t-t')\,\nu_k\, U_k(t,t') \\ &\quad + \theta(t-t')\int_{t_0}^{\infty} \mathrm{d}t''\,[\Sigma_k^{\mathrm{r}}(t,t'')\,U_k(t'',t') + \Sigma_k^{<}(t,t'')\,G_k(t',t'') + F_k(t,t'')\,G_k(t',t'')].\end{aligned} \tag{A 2}$$

This is then rewritten as

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\nu_k\right) U_k^{\mathrm{r}}(t,t') - \int_{t_0}^{\infty} \mathrm{d}t''\,\Sigma_k^{\mathrm{r}}(t,t'')\,U_k^{\mathrm{r}}(t'',t') \\ &\quad = U_k(t,t')\,\delta(t-t') + \theta(t-t')\int_{t_0}^{\infty} \mathrm{d}t''\,[\Sigma_k^{<}(t,t'')\,G_k(t',t'') + F_k(t,t'')\,G_k(t',t'')] \\ &\quad + \theta(t-t')\int_{t_0}^{\infty} \mathrm{d}t''\,\Sigma_k^{\mathrm{r}}(t,t'')\,U_k(t'',t') - \int_{t_0}^{\infty} \mathrm{d}t''\,\Sigma_k^{\mathrm{r}}(t,t'')\,U_k^{\mathrm{r}}(t'',t').\end{aligned} \tag{A 3}$$

Now note that the integral

$$\int_{t_0}^{\infty} \mathrm{d}t''\,\Sigma_k^{\mathrm{r}}(t,t'')\,U_k^{\mathrm{r}}(t'',t')$$

vanishes if $t' > t$ since the integrand then vanishes identically. Hence this integral can be rewritten as

$$\theta(t-t')\int_{t_0}^{\infty} \mathrm{d}t''\,\Sigma_k^{\mathrm{r}}(t,t'')\,U_k^{\mathrm{r}}(t'',t') = \theta(t-t')\int_{t'}^{\infty} \mathrm{d}t''\,\Sigma_k^{\mathrm{r}}(t,t'')\,U_k(t'',t'),$$

where the last line follows from the definition of $U_k^{\mathrm{r}}(t,t')$. We can thus combine the two integrals on the last line of (A 3) to obtain

$$\begin{aligned}&\left(\frac{\partial}{\partial t}+\nu_k\right) U_k^{\mathrm{r}}(t,t') - \int_{t_0}^{\infty} \mathrm{d}t''\,\Sigma_k^{\mathrm{r}}(t,t'')\,U_k^{\mathrm{r}}(t'',t') \\ &\quad = U_k(t,t')\,\delta(t-t') + \theta(t-t')\int_{t_0}^{t'} \mathrm{d}t''\,[\Sigma_k^{\mathrm{r}}(t,t'')\,U_k(t'',t') \\ &\quad + \Sigma_k^{<}(t,t'')\,G_k(t',t'') + F_k(t,t'')\,G_k(t',t'')],\end{aligned} \tag{A 4}$$

where we have used the retarded time dependence of G_k to define the upper limit on the last two integrals. The integro-differential operator acting on the left-hand side of (A 4) is exactly the same as in (2.5) and defines the inverse response function $G_k^{-1}(t,t')$. That is, (2.5) can be written as

$$\int_{t_0}^{\infty} \mathrm{d}t''\,G_k^{-1}(t,t'')\,G_k(t'',t') = \delta(t-t'), \tag{A 5}$$

where

$$G_k^{-1}(t,t'') = \left(\frac{\partial}{\partial t}+\nu k^2\right)\delta(t-t'') - \Sigma_k^{\mathrm{r}}(t,t'').$$

Thus, multiplying (A 4) by G_k and integrating, we obtain

$$\begin{aligned}U_k(t,t')\,\theta(t-t') &= G_k(t,t')\,U_k(t',t') \\ &\quad + \int_{t_0}^{\infty} \mathrm{d}t'' \int_{t_0}^{t'} \mathrm{d}t'''\,G_k(t,t'')\,\theta(t''-t')\,[\Sigma_k^{\mathrm{r}}(t'',t''')\,U_k(t''',t') \\ &\quad + \Sigma_k^{<}(t'',t''')\,G_k(t',t''') + F_k(t'',t''')\,G_k(t',t''')].\end{aligned} \tag{A 6}$$

This relationship simplifies in the case of white-noise forcing, for in that case the integral involving F_k vanishes identically due to the retarded time structure of $G_k(t,t')$. Thus for white-noise forcing we have

$$U_k(t,t')\,\theta(t-t') = G_k(t,t')\,U_k(t',t') + \int_{t'}^{t} dt'' \int_{t_0}^{t'} dt''' G_k(t,t'')\,[\Sigma_k^{\mathrm{r}}(t'',t''')\,U_k(t''',t') + \Sigma_k^{<}(t'',t''')\,G_k(t',t''')], \tag{3.4}$$

where we have made explicit the limits on the integrals.

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